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TRACE66

Trajectory Analysis and Orbit Determination Program,

Volume X: Lunar Gravity Analysis

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The TRACE66 Trajectory Analysis and Orbit Determination Program is a general-purpose orbital analysis program. It was written specifically for the CDC 6000 series computers to assist The Aerospace Corporation personnel in the analysis and design of satellite orbits and tracking systems. Volume X in its documentation series is a technical reference for the Lunar Gravity Field Analyzer of TRACE66. A comprehensive description is presented of the analysis procedure and of associated techniques used for dynamical determination of lunar gravitational constants and reconstruction of lunar satellite orbits.

The TRACE66 documentation series is summarized as follows:

Volume I:	General Program Objectives, Description and Summary
Volume II:	Coordinate & Time-Keeping Systems with Associated Transformations
Volume III:	Trajectory Generation Equations & Methods
Volume IV:	Measurement Data Generation & Observational Measurement Partials
Volume V:	Differential Correction Procedure and Techniques
Volume VI:	Orbital Statistics Via Covariance Analysis
Volume VII:	Usage Guide
Volume VIII:	Usage Overview

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Volume X:	Lunar Gravity Analysis
Volume XI:	LGA Data Processor
Volume XII:	Sequential Least Squares & Recursive Filter Procedures and Techniques

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TRACE66 TRAJECTORY ANALYSIS AND ORBIT DETERMINATION PROGRAM,
Volume X: Lunar Gravity Analysis

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THE AEROSPACE CORPORATION

Prepared for SPACE AND MISSILE SYSTEMS ORGANIZATION
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Los Angeles, California

FOREWORD

This report is published by The Aerospace Corporation, El Segundo, California, under Air Force Contract Nos. F04701-70-C-0059 and F04701-71-C-0172.

This report, which documents research carried out from July 1970 through July 1971, was submitted for review and approval on 5 August 1971 to Stuart R. Weinstein, Capt. USAF.

The Aerospace Corporation TRACE66 Trajectory Analysis and Orbit Determination Program has evolved during the past two years from the design and implementation efforts of many individuals. Analysis and programming contributions have been made by G. Buechler, E. H. Fletcher, R. J. Mercer, and A. J. Rusick. In addition, consultations with W. T. Kyner have led to many significant improvements and added capabilities within the program. The TRACE66 documentation series is, like the program's growth and development, a joint effort.

The authors wish to particularly acknowledge the numerous contributions of L. Wong; they also wish to recognize B. A. Troesch for this thorough criticism of this document, which resulted in many improvements.

Approved by

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Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

Stuart R. Weinstein Capt USAF

Stuart R. Weinstein, Capt. USAF

NOTICE

Some of the previously published volumes of the TRACE66 documentation series have been published by The Aerospace Corporation as Technical Operating Reports. Volume III: Trajectory Generation Equations and Methods was published as TOR-0066(9320)-2, Vol III; Volume V: Differential Correction Procedure and Techniques as TOR-0066(9320)-2, Vol V.

Volume VII: Usage Guide was published as TR-0059(9320)-1, Vol VII. Future volumes in this series will be published as Technical Reports.

ABSTRACT

The TRACE66 Trajectory Analysis and Orbit Determination Program is a general-purpose orbital analysis program. It was written specifically for the CDC 6000 series computers to assist The Aerospace Corporation personnel in the analysis and design of satellite orbits and tracking systems. Volume X in its documentation series is a technical reference for the Lunar Gravity Field Analyzer of TRACE66. A comprehensive description is presented of the analysis procedure and of associated techniques used for dynamical determination of lunar gravitational constants and reconstruction of lunar satellite orbits.

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1. INTRODUCTION

1.1 PURPOSE, SCOPE, AND LIMITATIONS

This is Volume X, Lunar Gravity Analysis, of a documentation series for the TRACE66 Trajectory Analyses and Orbit Determination Program. In particular, it is intended as a technical reference for the Lunar Gravity Field Analyzer of TRACE66. Significant analysis procedures and techniques used to determine lunar gravitational constants and reconstruct lunar satellite orbits are described.

The symbols and notations of coordinate system transformations associated with reference frames and equations throughout the document are defined and discussed in Section 2. Equations of motion and variational equations are expressed in Section 3, with emphasis on the description of each model used to evaluate the various accelerations acting on the vehicle.

In Section 4, the observation measurement model for long-count two-way doppler data is described. Sections 5 and 6 present, respectively, the details associated with the accumulation of the normal matrix, and the solution of the normal equations.

Techniques used to obtain a composite lunar gravity field are described in Section 7, while features peculiar to the TRACE66 Lunar Gravity Field Analyzer are discussed in Section 8.

1.2 HISTORICAL BACKGROUND

The fundamental purpose of TRACE is to simulate orbital motion and tracking operations. In the past, lunar orbit determination was performed at The Aerospace Corporation by the TRACE-DL Orbit Determination Program (Ref. 1). The current TRACE66 program has significant capabilities that surpass those of TRACE-DL.

1.3 TECHNICAL BACKGROUND

In particular, this document summarizes the software developed for the dynamical determination of lunar gravitational constants and reconstruction of lunar satellite orbits. Recent analyses and investigations (Refs. 2 and 3) of NASA Deep Space Network doppler observations of Lunar Orbiters have utilized TRACE66 and auxiliary software for dynamical fitting and composite model determination.

1.4 SUMMARY

This document describes both the TRACE66 Lunar Gravity Field Analyzer and external auxiliary software that determine lunar gravitational constants and reconstruct lunar satellite orbits from NASA Deep Space Network 2-way doppler data. The force model used to generate lunar trajectories is a combination of perturbative accelerations resulting from

- Gravitational potential due to a surface distribution grid (750 points) of disk or point masses $\{\mu_i, \phi_i, \lambda_i, a_i\}$
- Lunar gravitational potential $\{\mu, C_{nm}, S_{nm}\}$
- Gravitational potential due to planetary bodies (e.g., Sun and Earth).

Principal features of interest here are the following capabilities:

- Mass parameter selection by orbital arc
- Doppler residual generation with printer plot versus vehicle selenographic ground track

- Residual and edit summaries
- Self-contained orbit determination function if the number of parameters does not exceed 100
- Normal matrix (256 × 256) accumulation per arc (exterior to TRACE66, these normal matrices are merged and the resulting linear system, of the order 2000, is solved).

2. SYMBOLS AND NOTATION

The common symbols used throughout this document are presented next, followed by a discussion of the basic notation associated with coordinate system transformations.

2.1 SYMBOLS

$\cdot$	time differentiation and vector dot product operation
$\cdot\cdot$	time differentiation
_	as an underscore, indicates a vector; for example, $\underline{r}$ is the vehicle position vector
Δ	increment or difference
$\times$	vector cross-product operation and product sign
| |	absolute value, or vector magnitude operation; for example, $r = \lvert\underline{r}\rvert$
∇	vector gradient operator
( )	indicates a functional relationship or dependence, such as the vehicle acceleration vector $\underline{\ddot{r}}(\underline{r}, \underline{\dot{r}}, \gamma)$, which is a function or $\underline{r}, \underline{\dot{r}}$, and γ. Also, used to denote row vectors; for example, $\underline{r} = (x, y, z)$
{ }	indicates a set of elements; for example, $\{\mu, \rho, \Phi, \lambda\}$
[ ]	a matrix; for example, $\underline{r} = [x, y, z]$ (row vector) or

$$\underline{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{(column vector)}$$

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$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ 3 × 3 identity matrix

Superscripts

i, j	particular index relative to a set or vector
c	computed value
T	matrix transpose symbol
n	iteration index

Subscripts

0, 1, 2, 3	index denoting forces, components, or axis of rotation
i, j, k, ℓ	particular index denoting members of a set or components of a vector. In general, i, j, k, ℓ = 1, 2, . . .
M	related to the Moon
c, mc	computed value, and measured minus computed value
T, C	transmitter and count time
ℓh	local horizon reference frame
nm	degree n and order m
m, n	denotes index
x, y, z	coordinate axes

2.2 COORDINATE SYSTEM TRANSFORMATIONS

Lunar gravity analysis requires the consideration of several reference coordinate systems (see Volume II). The basic transformations of interest here relate one system to another; they are described by the following examples:

In the Moon-Centered Inertial (MCI) relationship to the selenographic or Moon Fixed (MF) frame

$$\underline{r}_{TEE}^{MF} = [M]\, \underline{r}_{MEE}^{MCI}$$

where

$\underline{r}_{MEE}^{MCI}$ = MCI position vector of the vehicle referenced to a mean equator and equinox of some base date

$[M]$ = time-dependent transformation matrix from a mean rectangular inertial frame of base date to a true selenographic (moon fixed) frame

$[M] = [S]\,[P]$

$= [SP]$

$[P]$ = time-dependent <u>precession</u> matrix, which transforms from mean equator and equinox of some base date to mean equator and equinox of date (see Figures 1 and 2).

$[S]$ = time-dependent transformation matrix from mean rectangular inertial of date to true selenographic of date

$\underline{r}_{TEE}^{MF}$ = MF position vector of the vehicle referenced to a true equator and equinox of date.

In the selenographic or Moon-Fixed (MF) relationship to the Moon-Centered Inertial (MCI)

$$\ddot{\underline{r}}^{MCI}_{MEE} = [M^{-1}] \; \ddot{\underline{r}}^{MF}_{TEE}$$

where

$[\dot{M}] = [\ddot{M}] \equiv 0$ (by assumption)

$\ddot{\underline{r}}^{MF}_{TEE}$ = MF acceleration vector of the vehicle referenced to a true equator and equinox of date

$[M]^{-1}$ = time-dependent transformation matrix from a true selenographic frame to a mean rectangular inertial frame; note that

$$[M^{-1}] = [SP]^{-1}$$
$$= [P]^{-1} [S]^{-1}$$

$\ddot{\underline{r}}^{MCI}_{MEE}$ = MCI acceleration vector of the vehicle referenced to a mean equator and equinox of some base date (this is the coordinate and time-keeping system in the chosen integration system).

In the Moon-Centered Inertial (MCI) relationship to the Earth-Centered Inertial (ECI):

$$\underline{r}^{ECI} = \underline{r}^{MCI} + \underline{r}^{ECI}_{M}$$

where

$\underline{r}^{ECI}$ = ECI position vector of the vehicle

$\underline{r}^{MCI}$ = MCI position vector of the vehicle

$\underline{r}_M^{ECI}$ = ECI position vector of the Moon

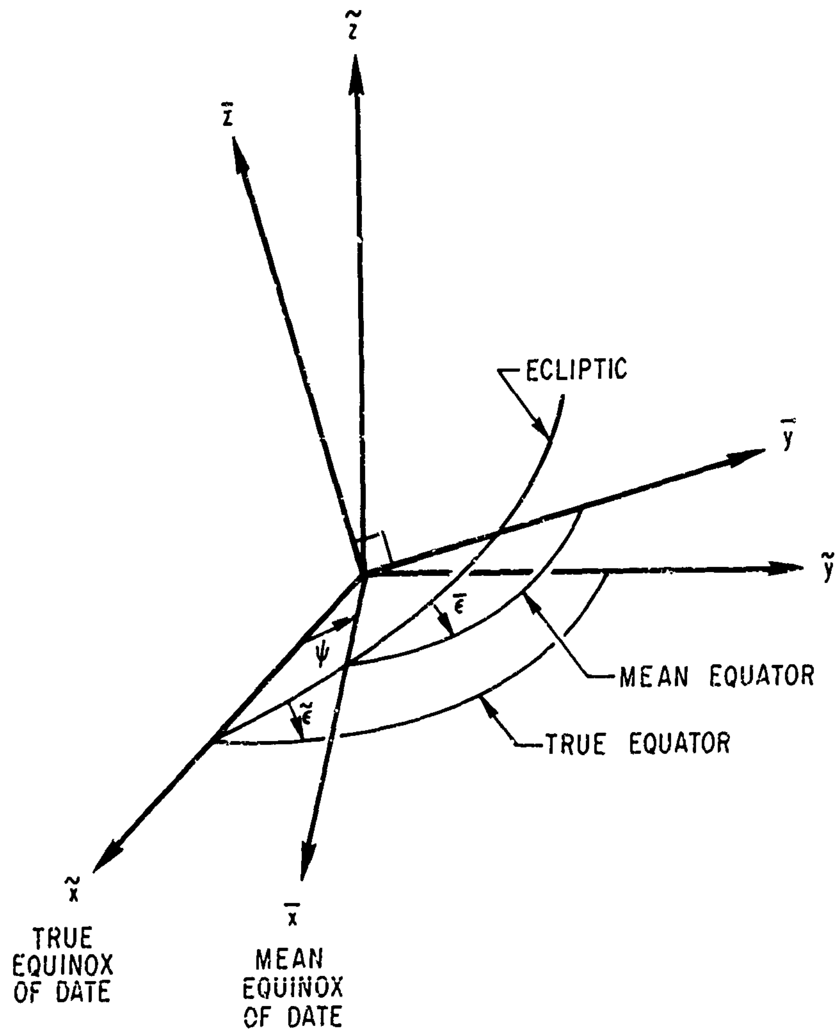

Figure 1. The Mean and True Equator and Equinox of Date

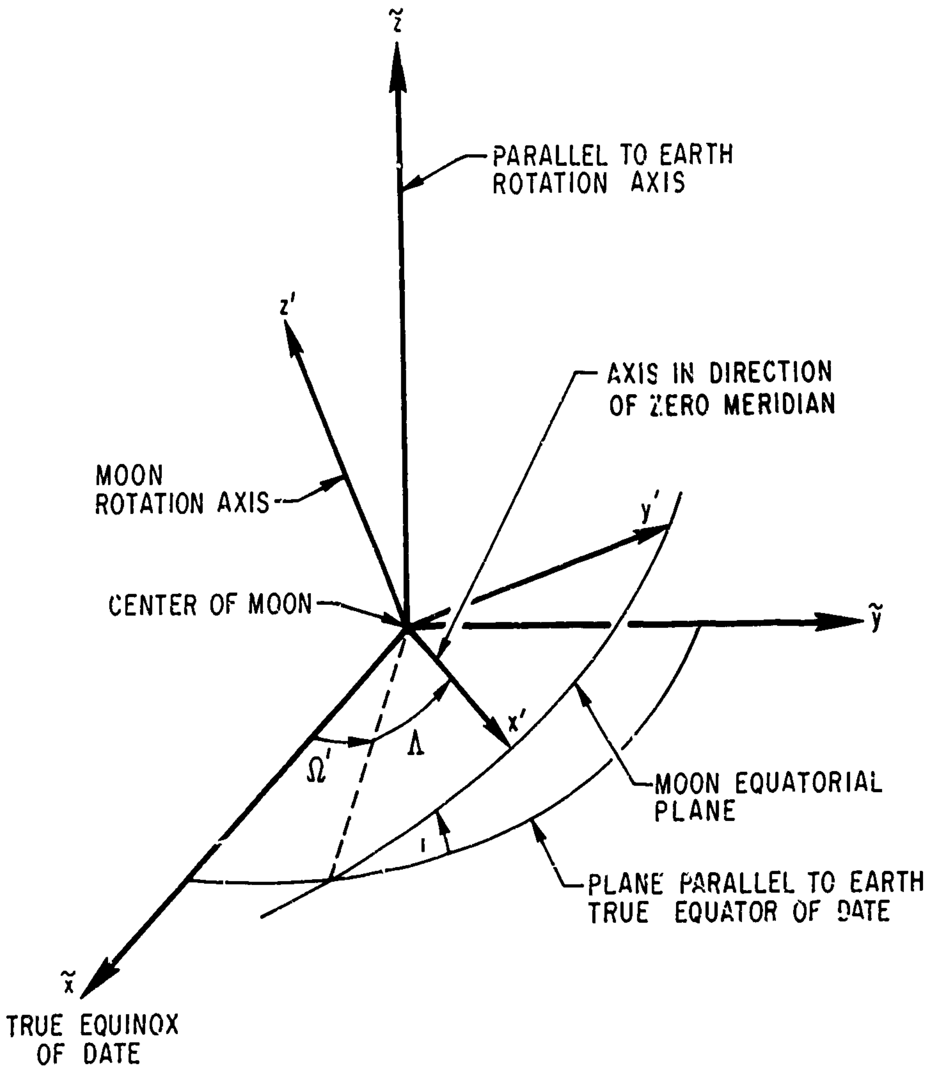

Figure 2. The Moon-Fixed System and the True Equator and Equinox of Date

3. EQUATIONS OF MOTION AND VARIATIONAL EQUATIONS

In the TRACE66 Lunar Analyzer, the vector equation of motion is

$$\underline{\ddot{r}} = \sum_{i=0}^{2} \underline{\ddot{r}}_i \tag{1}$$

where

$\underline{\ddot{r}}$ = total acceleration of vehicle

$\underline{\ddot{r}}_0$ = acceleration due to <u>discrete masses</u> imbedded in the Moon

$\underline{\ddot{r}}_1$ = gravitational acceleration due to the <u>Moon</u>

$\underline{\ddot{r}}_2$ = gravitational acceleration due to the <u>Sun and Earth</u>

Equation (1) is numerically integrated in a Cowell formulation with an 8th-order predictor/corrector Gauss-Jackson differencing method.

Each acceleration considered is evaluated in its appropriate reference frame. If necessary, it is rotated and accumulated in a moon-centered inertial (MCI) frame, which is then used as the integration coordinate system (see Volume II).

The variational equation for any equation-of-motion parameter γ (Ref. 3) is expressed by the vector differential equation

$$\frac{\partial \underline{\ddot{r}}}{\partial \gamma} = \left[\frac{\partial \underline{\ddot{r}}}{\partial \underline{r}}\right] \frac{\partial \underline{r}}{\partial \gamma} + \left[\frac{\partial \underline{\ddot{r}}}{\partial \underline{\dot{r}}}\right] \frac{\partial \underline{\dot{r}}}{\partial \gamma} + \frac{\partial \underline{\ddot{r}}(\underline{r}, \underline{\dot{r}}, \gamma)}{\partial \gamma} \tag{2}$$

Partial derivative matrices $\partial\underline{\ddot{r}}/\partial\underline{r}$ and $\partial\underline{\ddot{r}}/\partial\underline{\dot{r}}$ are obtained by differentiating the total acceleration with respect to $\underline{r}$ and $\underline{\dot{r}}$. The first two terms of the

three in Eq. (2) account for the implicit appearance of γ through the functional dependence of $\underline{r}$ and $\dot{\underline{r}}$. The last term is referred to as the nonhomogeneous term, and accounts for the explicit appearance of γ in the total acceleration. Because γ is independent of time and the derivatives are continuous, the order of differentiation in Eq. (2) may be interchanged. A double integration with respect to time then yields the trajectory partial derivatives $\partial\underline{r}/\partial\gamma$.

In the following three sections, the models used in evaluating the accelerations of Eq. (1) are presented in detail; also, the partial derivative matrices $[\partial\ddot{\underline{r}}/\partial\underline{r}]$ and $[\partial\ddot{\underline{r}}/\partial\dot{\underline{r}}]$, and nonhomogeneous terms for the variational equations, are presented where applicable. In Section 3.4, the use of partial derivatives is discussed.

3.1 ACCELERATION DUE TO DISCRETE MASSES

In TRACE66, a discrete mass force model can be represented by a point or disk mass formulation.

3.1.1 Point Mass Acceleration

Acceleration due to point masses imbedded in the Moon is given by

$$\underline{\ddot{r}}_0 = -\mu \sum_i \mu_i \frac{\underline{r} - [M^{-1}]\underline{P}_i}{|\underline{r} - [M^{-1}]\underline{P}_i|^3} \tag{3}$$

where

$\underline{P}_i \equiv$ the selenographic cartesian position vector of the i^{th} point mass

$\mu_i \equiv$ the mass ratio of the i^{th} point relative to the Moon

$\mu \equiv$ the gravitational constant of the Moon

$[M] \equiv$ the transformation matrix from the mean selenocentric frame of base date to the true selenographic frame

$\underline{r} \equiv$ the selenocentric (MCI) position vector of the vehicle where $\underline{r} = (x, y, z)^T$

The selenographic position vector $\underline{P}_i$ of the i^{th} point mass is determined by the triplet $(\rho_i, \Phi_i, \lambda_i)$, where ρ_i is the selenocentric distance of the point and Φ_i, λ_i are the selenographic latitude and longitude of the point mass. In particular,

$$\underline{P}_i = (\rho_i \cos \Phi_i \cos \lambda_i, \rho_i \cos \Phi_i \sin \lambda_i, \rho_i \sin \Phi_i)^T \tag{4}$$

The mass ratio μ_i is the differentially correctable point mass parameter. The variational equations for μ_i, ρ_i, Φ_i, λ_i, have the nonhomogeneous terms

$$\frac{\partial \ddot{\underline{r}}_0}{\partial \mu_i} = -\mu \frac{\underline{r} - [M^{-1}]\,\underline{P}_i}{|\underline{r} - [M^{-1}]\,\underline{P}_i|^3}$$

$$\frac{\partial \ddot{\underline{r}}_0}{\partial \rho_i} = \frac{\mu\mu_i[M^{-1}]}{|\underline{r} - [M^{-1}]\underline{P}_i|^3}\left[\frac{\partial \underline{P}_i}{\partial \rho_i} - 3\left[(\underline{r} - [M^{-1}]\underline{P}_i)\cdot\frac{\partial \underline{P}_i}{\partial \rho_i}\right]\frac{\underline{r} - [M^{-1}]\underline{P}_i}{|\underline{r} - [M^{-1}]\underline{P}_i|^2}\right]$$

$$\frac{\partial \ddot{\underline{r}}_0}{\partial \Phi_i} = \frac{\mu\mu_i[M^{-1}]}{|\underline{r} - [M^{-1}]\underline{P}_i|^3}\left[\frac{\partial \underline{P}_i}{\partial \Phi_i} - 3\left[(\underline{r} - [M^{-1}]\underline{P}_i)\cdot\frac{\partial \underline{P}_i}{\partial \Phi_i}\right]\frac{\underline{r} - [M^{-1}]\underline{P}_i}{|\underline{r} - [M^{-1}]\underline{P}_i|^2}\right]$$

$$\frac{\partial \ddot{\underline{r}}_0}{\partial \lambda_i} = \frac{\mu\mu_i[M^{-1}]}{|\underline{r} - [M^{-1}]\underline{P}_i|^3}\left[\frac{\partial \underline{P}_i}{\partial \lambda_i} - 3\left[(\underline{r} - [M^{-1}]\underline{P}_i)\cdot\frac{\partial \underline{P}_i}{\partial \lambda_i}\right]\frac{\underline{r} - [M^{-1}]\underline{P}_i}{|\underline{r} - [M^{-1}]\underline{P}_i|^2}\right] \tag{5}$$

where

$$\frac{\partial \underline{P}_i}{\partial \rho_i} = (\cos\Phi_i \cos\lambda_i,\ \cos\Phi_i \sin\lambda_i,\ \sin\Phi_i)^T$$

$$\frac{\partial \underline{P}_i}{\partial \Phi_i} = (-\rho_i \sin\Phi_i \cos\lambda_i,\ -\rho_i \sin\Phi_i \sin\lambda_i,\ \rho_i \cos\Phi_i)^T$$

$$\frac{\partial \underline{P}_i}{\partial \lambda_i} = (-\rho_i \cos\Phi_i \sin\lambda_i,\ \rho_i \cos\Phi_i \cos\lambda_i,\ 0)^T$$

The derivatives of the point mass acceleration with respect to the vehicle position $\underline{r}$ and velocity $\underline{\dot{r}}$ are given by the matrices

$$\frac{\partial \underline{\ddot{r}}_0}{\partial \underline{r}} = -\mu \sum_i \frac{\mu_i}{|\underline{r} - [M^{-1}]\underline{P}_i|^3}\left[I - 3\frac{\left(\underline{r} - [M^{-1}]\underline{P}_i\right)\left(\underline{r} - [M^{-1}]\underline{P}_i\right)^T}{|\underline{r} - [M^{-1}]\underline{P}_i|^2}\right] \tag{6}$$

and

$$\frac{\partial \underline{\ddot{r}}_0}{\partial \underline{\dot{r}}} = [0] \tag{7}$$

where $I = 3 \times 3$ identity matrix.

3.1.2 Disk Mass Acceleration

Acceleration due to disk masses is formulated as one resulting from a limiting form of zero-thickness disks (Refs. 4 and 5).

The geometry used to relate the disk masses to the lunar center is given in Figure 3.

The disks are considered to be infinitesimally thin, planar, and usually located near the surface of the Moon, the mean equatorial radius of which is a_m. Now by letting $\underline{n}$ be normal to the sphere, coinciding with the axis of the disk, the local horizon system $(x_{\ell h}, y_{\ell h}, z_{\ell h})$ has its origin at the center of the disk where: $z_{\ell h}$ is parallel to $\underline{n}$; $x_{\ell h}$ is directed south on the plane normal to $\underline{n}$; and $y_{\ell h}$ is east on the same plane. The selenographic system (x, y, z) origin is located at the center of the Moon, where z is along the polar axis, and x is in the direction of the intersection of the equatorial plane and the zero meridian.

The position and acceleration vectors in the local horizon system are expressed as

$$\underline{r}_{\ell h} = (x_{\ell h}, y_{\ell h}, z_{\ell h})^T$$
$$\ddot{\underline{r}}_{\ell h} = (\ddot{x}_{\ell h}, \ddot{y}_{\ell h}, \ddot{z}_{\ell h})^T \tag{8}$$

The selenographic position vector $\underline{P}_i$ of the i^{th} disk mass is the same as Eq. (4). Therefore, the transformation of an arbitrary selenographic vector $\underline{r}_{MF}$ to the local horizon frame specified by the i^{th} disk mass is given by

$$\underline{r}_{\ell h} = R_2\left(\frac{\pi}{2} - \phi\right) R_3(\lambda) \underline{r}_{MF} - [C] \tag{9}$$

where

$$R_2\left(\frac{\pi}{2} - \phi\right) = \begin{bmatrix} \sin\phi & 0 & -\cos\phi \\ 0 & 1 & 0 \\ \cos\phi & 0 & \sin\phi \end{bmatrix}$$

$$R_3(\lambda) = \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{C} = \begin{bmatrix} 0 & 0 & \rho \end{bmatrix}^T$$

and ϕ, λ are the latitude and longitude of the mass disk, respectively.

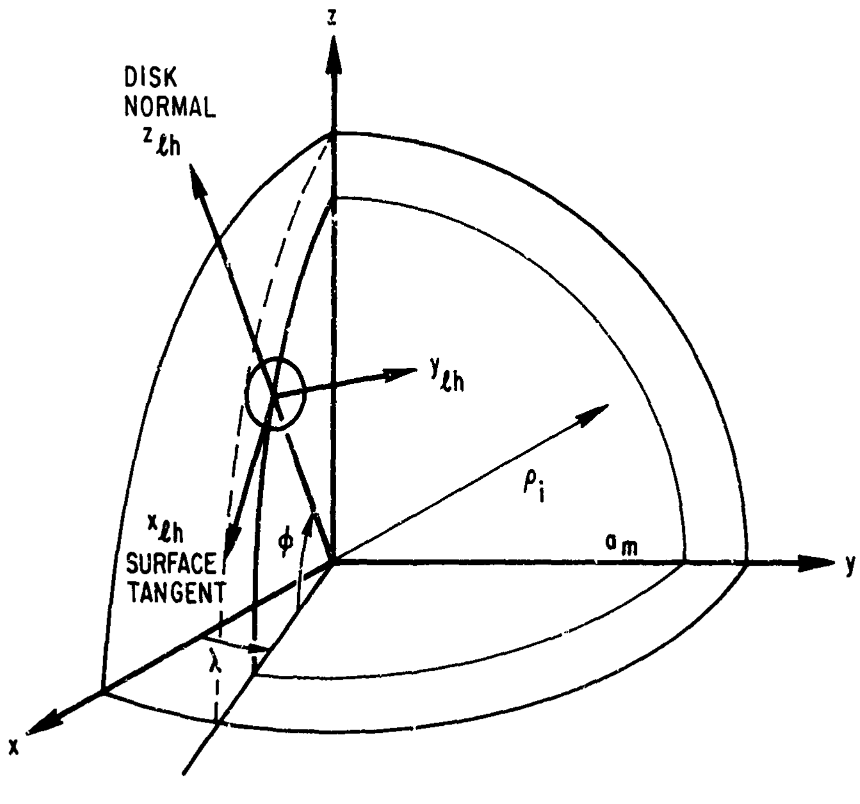

Figure 3. Disk Mass Geometry

The components of the acceleration vector due to the mass disk in the local horizon system are

$$\frac{\ddot{x}_{\ell h}}{x_{\ell h}} = \frac{\ddot{y}_{\ell h}}{y_{\ell h}} = \frac{3}{2} \frac{\mu}{a^3} \mu_i \left[\frac{a\sqrt{K}}{a^2+K} - \tan^{-1} \frac{a}{\sqrt{K}} \right]$$

$$\frac{\ddot{z}_{\ell h}}{z_{\ell h}} = \frac{3}{2} \frac{\mu}{a^3} \mu_i \left[\tan^{-1} \frac{a}{\sqrt{K}} - \frac{a}{\sqrt{K}} \right] \qquad (10)$$

where

$\mu \equiv$ the gravitational constant of the Moon

$\mu_i \equiv$ the mass magnitude of the i^{th} disk

$a \equiv$ the radius of the disk

By solving the quadratic equation (Ref. 3)

$$K^2 + (a^2 - R^2)K - z_{\ell h}^2 a^2 = 0 \qquad (11)$$

for K, and choosing the positive root, its value is

$$K = \frac{1}{2} \left[(R^2 - a^2) + \sqrt{(R^2 - a^2)^2 + 4 z_{\ell h}^2 a^2} \right] \qquad (12)$$

where

$$R^2 = |[M] \underline{r} - \underline{P}_i|^2$$

and where $\underline{r}$ is the selenocentric position vector of the vehicle. The quantity $[M]$ is the transformation matrix from the mean selenocentric frame to the selenographic frame.

Transformation of the above accelerations to the selenographic frame is defined as

$$\ddot{\underline{D}}_i = R_3^T(\lambda)\, R_2^T\left(\frac{\pi}{2} - \phi\right) \ddot{\underline{r}}_{\ell h} \tag{13}$$

Thus, acceleration due to disk masses imbedded in the Moon becomes

$$\ddot{\underline{r}}_0 = \left[M^{-1}\right] \sum_i \ddot{\underline{D}}_i \tag{14}$$

The differentially correctable disk mass parameter is the mass ratio μ_i, the variational equation for which has the nonhomogeneous term

$$\frac{\partial \ddot{\underline{r}}_0}{\partial \mu_i} = \left[M^{-1}\right] \frac{\partial \ddot{\underline{D}}_i}{\partial \mu_i} \tag{15}$$

where

$$\frac{\partial \ddot{\underline{D}}_i}{\partial \mu_i} = R_3^T(\lambda)\, R_2^T\left(\frac{\pi}{2} - \phi\right) \frac{\partial \ddot{\underline{r}}_{\ell h}}{\partial \mu_i}$$

$$\frac{\partial \ddot{\underline{r}}_{\ell h}}{\partial \mu_i} = \frac{3}{2}\mu\, a^{-3} \begin{bmatrix} \left[(aK^{1/2})/(a^2+K) - \tan^{-1}(aK^{-1/2})\right] x_{\ell h} \\ \left[(aK^{1/2})/(a^2+K) - \tan^{-1}(aK^{-1/2})\right] y_{\ell h} \\ \left[\tan^{-1}(aK^{-1/2}) - (aK^{-1/2})\right] z_{\ell h} \end{bmatrix}$$

The derivatives of the disk mass acceleration with respect to the vehicle position $\underline{r}$ and velocity $\dot{\underline{r}}$ are given by the matrices

$$\frac{\partial \ddot{\underline{r}}_0}{\partial \underline{r}} = -\mu \sum \frac{\mu_i}{|\underline{r} - [M^{-1}]\underline{P}_i|^3} \left[I - 3 \frac{\left(\underline{r} - [M^{-1}]\underline{P}_i\right)\left(\underline{r} - [M^{-1}]\underline{P}_i\right)^T}{|\underline{r} - [M^{-1}]\underline{P}_i|^2} \right] \quad (16)$$

and

$$\frac{\partial \ddot{\underline{r}}_0}{\partial \dot{\underline{r}}} = [0] \quad (17)$$

where I is a 3 × 3 matrix.

3.2 GRAVITATIONAL ACCELERATION DUE TO THE MOON

Acceleration due to the Moon is computed from the gradient of the potential function

$$U = \frac{\mu}{r}\left[1 + \sum_{n=1}^{\infty}\left(\frac{a_m}{r}\right)^n \sum_{m=0}^{n} P_n^m (\sin\delta)\left(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right)\right] \tag{18}$$

where

μ = gravitational constant GM of the Moon

r, δ, λ = selenographic distance, latitude, and East longitude of the vehicle, respectively

a_m = mean equatorial radius of the Moon

P_n^m = Legendre associated function of the first kind, of degree n and order m

C_{nm}, S_{nm} = numerical coefficients.

The gradient of the potential U is computed in the radial, east, and north directions of the local horizontal coordinate system. Thus

$$\ddot{\underline{r}}_1 = \left[M^{-1}\right][R]\nabla U \tag{19}$$

where ∇ is the gradient operator, and $\left[M^{-1}\right]$ is the rotation matrix from the selenographic (MF true equator) frame to the selenocentric (MCI mean equator and equinox) frame. Thus

$$\begin{aligned}[M] &= [SP] \\ &= [S][P]\end{aligned} \tag{20}$$

where

$[P] = [P(\zeta_o, Z, \theta)]$ precession

where

$-\zeta_o$ = right ascension of the mean celestial pole of date, referred to the mean equator and equinox at the base date

$\pi+Z$ = right ascension of the mean celestial pole at the base date, referred to the mean equator and equinox of date

$\theta = \pi/2$ - declination (i.e., the north polar distance) of the mean celestial pole of date, referred to the mean equator at the base date

and

$[S] = [S(\Omega', \Lambda, i, \psi, \tilde{\epsilon}, \bar{\epsilon})]$

where

Ω', Λ, i = lunar Eulerian angles

ψ = nutation in longitude

$\tilde{\epsilon}$ = true obliquity of the ecliptic to the Earth's equator

$\bar{\epsilon}$ = mean obliquity of the ecliptic to the Earth's equator.

Note that [M] (see Ref. 6) transforms the mean equator from MCI to the true MF coordinates. The rotation matrix [R] in Eq. (19) is defined as

$$[R] = \begin{bmatrix} \cos\delta\cos\lambda & -\sin\lambda & -\sin\delta\cos\lambda \\ \cos\delta\sin\lambda & \cos\lambda & -\sin\delta\sin\lambda \\ \sin\delta & 0 & \cos\delta \end{bmatrix}$$

which is the transformation from the local horizontal frame to the selenographic frame.

The components of the ∇U are

$$g_1 = \frac{\partial U}{\partial r} = \frac{-\mu}{r^2}\left[1 + \sum_{n=1}^{\infty}(n+1)\left(\frac{a_m}{r}\right)^n \sum_{m=0}^{n} P_n^m(\sin\delta)(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\right]$$

$$g_2 = \frac{1}{r\cos\delta}\frac{\partial U}{\partial\lambda}$$

$$= \frac{-\mu}{r^2\cos\delta}\left[\sum_{n=1}^{\infty}\left(\frac{a_m}{r}\right)^n \sum_{m=0}^{n} mP_n^m(\sin\delta)(C_{nm}\sin m\lambda - S_{nm}\cos m\lambda)\right]$$

$$g_3 = \frac{1}{r}\frac{\partial U}{\partial\delta}$$

$$= \frac{\mu}{r^2}\left[\sum_{n=1}^{\infty}\left(\frac{a_m}{r}\right)^n \sum_{m=0}^{n} P_n^{m\prime}(\sin\delta)\cos\delta\,(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\right] \qquad (21)$$

where $P_n^{m\prime}(\sin\delta)$ is the derivative of the Legendre function with respect to $\sin\delta$, and subscripts 1, 2, 3 refer to radial, east, and north directions.

The recursive formulas used in computing the Legendre associated functions and their derivatives are divided into two groups. The terms with $m = 0$ (zonal harmonics) are computed by the recursive formulas

$$P_n^0 \equiv P_n = [(2n-1)\sin\delta\, P_{n-1} - (n-1)P_{n-2}]/n$$

and

$$P_n' \cos\delta = \sin\delta\cos\delta\, P_{n-1}' + n\cos\delta\, P_{n-1}$$

with the initial values $P_0 = P_1' = 1$ and $P_1 = \sin\delta$.

The terms with $m \neq 0$ (tesseral harmonics) are computed by the recursive formulas

$$P_n^m/\cos\delta = \left[(2n-1)\sin\delta\, P_{n-1}^m/\cos\delta - (n+m-1)\, P_{n-2}^m/\cos\delta\right]/(n-m)$$

and

$$P_n^{m'}\cos\delta = (n+m)\, P_{n-1}^m/\cos\delta - n\sin\delta\, P_n^m/\cos\delta$$

with the initial values $P_{m-1}^m = 0$, and $P_m^m/\cos\delta = 1\cdot 3 \cdots (2m-1)\cos^{m-1}\delta$.

There are two normalizations possible in TRACE66 (Ref. 7) when the C_{nm} and S_{nm} coefficients are specified. The unnormalized form is used in the above potential expression.

Variational equations for the differentially correctable gravitational parameters μ, C_{nm}, and S_{nm} are obtained by differentiating the relations in Eq. (21). These derivatives constitute the nonhomogeneous terms of the μ, C_{nm}, and S_{nm} variational equations

$$\frac{\partial g_1}{\partial \mu} = \frac{-1}{r^2}\left[1 + \sum_{n=1}^{\infty}(n+1)\left(\frac{a_m}{r}\right)^n \sum_{m=0}^{n} P_n^m (\sin\delta)(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\right]$$

$$\frac{\partial g_2}{\partial \mu} = \frac{-1}{r^2\cos\delta}\left[\sum_{n=1}^{\infty}\left(\frac{a_m}{r}\right)^n \sum_{m=0}^{n} m P_n^m (\sin\delta)(C_{nm}\sin m\lambda - S_{nm}\cos m\lambda)\right]$$

$$\frac{\partial g_3}{\partial \mu} = \frac{1}{r^2}\left[\sum_{n=1}^{\infty}\left(\frac{a_m}{r}\right)^n \sum_{m=0}^{n} P_n^{m\prime} (\sin\delta)\cos\delta\,(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\right]$$

$$\frac{\partial g_1}{\partial C_{nm}} = \frac{-\mu}{r^2}\left[(n+1)\left(\frac{a_m}{r}\right)^n P_n^m(\sin\delta)\cos m\lambda\right]$$

$$\frac{\partial g_2}{\partial C_{nm}} = \frac{-\mu}{r^2\cos\delta}\left[\left(\frac{a_m}{r}\right)^n m P_n^m(\sin\delta)\sin m\lambda\right]$$

$$\frac{\partial g_3}{\partial C_{nm}} = \frac{\mu}{r^2}\left[\left(\frac{a_m}{r}\right)^n P_n^{m\prime}(\sin\delta)\cos\delta\cos m\lambda\right]$$

$$\frac{\partial g_1}{\partial S_{nm}} = \frac{-\mu}{r^2}\left[(n+1)\left(\frac{a_m}{r}\right)^n P_n^m(\sin\delta)\sin m\lambda\right]$$

$$\frac{\partial g_2}{\partial S_{nm}} = \frac{\mu}{r^2\cos\delta}\left[\left(\frac{a_m}{r}\right)^n m P_n^m(\sin\delta)\cos m\lambda\right]$$

$$\frac{\partial g_3}{\partial S_{nm}} = \frac{\mu}{r^2}\left[\left(\frac{a_m}{r}\right)^n P_n^{m\prime}(\sin\delta)\cos\delta\sin m\lambda\right] \tag{22}$$

Note that the nonhomogeneous terms in Eq. (22) have the general form

$$\frac{\partial \underline{\ddot{r}}_1}{\partial \gamma} = [R] \frac{\partial \nabla U}{\partial \gamma} \tag{23}$$

where

$$\nabla U = [g_1, g_2, g_3]^T$$

$$\gamma = \mu, C_{nm}, \text{ or } S_{nm}$$

The derivatives of $\underline{\ddot{r}}_1$ with respect to the vehicle position $\underline{r}$ and velocity $\underline{\dot{r}}$ are the matrices

$$\begin{aligned} \frac{\partial \underline{\ddot{r}}_1}{\partial \underline{r}} &= [R] \frac{\partial \nabla U}{\partial \underline{g}} \frac{\partial \underline{g}}{\partial \underline{u}} \frac{\partial \underline{u}}{\partial \underline{r}} [R]^T \\ &= [R][H][R]^T \end{aligned} \tag{24}$$

where $\underline{g} = (g_1, g_2, g_3)^T$ is the acceleration in the local horizontal system, and $\underline{u}$ is the position vector in the horizontal system with the partial derivative matrix H given by

$$H = \frac{1}{r} \begin{bmatrix} r \dfrac{\partial g_1}{\partial r} & \dfrac{\partial g_1/\partial \lambda - g_2}{\cos \delta} & \dfrac{\partial g_1}{\partial \delta} - g_3 \\ r \dfrac{\partial g_2}{\partial r} & \dfrac{\partial g_2/\partial \lambda + g_1 - \tan \delta \, g_3}{\cos \delta} & \dfrac{\partial g_2}{\partial \delta} \\ \dfrac{\partial g_3}{\partial r} & \dfrac{\partial g_2/\partial \lambda}{\cos \delta} & \dfrac{\partial g_3}{\partial \delta} + g_1 \end{bmatrix} \tag{25}$$

and

$$\frac{\partial \underline{\ddot{r}}_1}{\partial \underline{\dot{r}}} = [0].$$

3.3 PLANETARY GRAVITATIONAL ACCELERATIONS

Gravitational acceleration due to planetary bodies is given by

$$\ddot{\underline{r}}_2 = -\mu \sum_i \mu_i \left[\frac{\underline{r}-\underline{r}_i}{|\underline{r}-\underline{r}_i|^3} + \frac{\underline{r}_i}{|\underline{r}_i|^3} \right] \tag{26}$$

where

μ_i = mass ratio of the i^{th} body relative to the Moon

$\underline{r}_i$ = MCI position vector of the i^{th} perturbing body

$\underline{r}$ = MCI position vector of the vehicle

Note that

$$\underline{r}_i = \underline{r}_i^{ECI} - \underline{r}_M^{ECI}$$

where

$\underline{r}_i^{ECI}$ = Earth-centered inertial (ECI) position vector of the i^{th} perturbing body

$\underline{r}_M^{ECI}$ = ECI position vector of the Moon

[Ephemerides are obtained by making the necessary coordinate frame rotation and interpolating a planetary ephemeris file created from a Jet Propulsion Laboratory master tape (Ref. 8)].

There are no differentially correctable planetary gravitational parameters.

With respect to vehicle position $\underline{r}$ and velocity $\dot{\underline{r}}$, the derivative matrices of $\ddot{\underline{r}}_2$ are

$$\frac{\partial \ddot{\underline{r}}_2}{\partial \underline{r}} = -\mu \sum_i \frac{\mu_i}{|\underline{r}-\underline{r}_i|^3} \left[I - \frac{3}{|r-r_i|^2} (r-r_i)(r-r_i)^T \right] \tag{27}$$

where I = the 3×3 identity matrix, and $(\partial \ddot{\underline{r}}_2)/(\partial \dot{\underline{r}}) = [0]$.

3.4 PARTIAL DERIVATIVES

If O_c denotes the vector of computed measurements and P the parameter vector, the measurement partial derivatives are

$$\frac{\partial O_c}{\partial P} = \frac{\partial O_c}{\partial \underline{r}}\left(\frac{\partial \underline{r}}{\partial P}\right) + \frac{\partial O_c}{\partial \underline{\dot{r}}}\left(\frac{\partial \underline{\dot{r}}}{\partial P}\right) \tag{28}$$

if the components of P are equation-of-motion-dependent parameters. Then

$\left.\dfrac{\partial O_c}{\partial \underline{r}}\ ,\ \dfrac{\partial O_c}{\partial \underline{\dot{r}}}\right\}$ are called geometric measurement partials

$\left.\left(\dfrac{\partial \underline{r}}{\partial P}\right)\ ,\ \left(\dfrac{\partial \underline{\dot{r}}}{\partial P}\right)\right\}$ are called trajectory partials, which are solutions of the variational equations

Note that for measurements obtained from Earth-based sensors, TRACE66 considers

$$\begin{aligned}\frac{\partial m_c^j}{\partial p_i} &= \frac{\partial m_c^j}{\partial \underline{r}_{ECI}}\,\frac{\partial \underline{r}_{ECI}}{\partial \underline{r}_{MCI}}\left(\frac{\partial \underline{r}_{MCI}}{\partial p_i}\right) \\ &+ \frac{\partial m_c^j}{\partial \underline{\dot{r}}_{ECI}}\,\frac{\partial \underline{\dot{r}}_{ECI}}{\partial \underline{\dot{r}}_{MCI}}\left(\frac{\partial \underline{\dot{r}}_{MCI}}{\partial p_i}\right)\end{aligned} \tag{29}$$

where

$$\frac{\partial \underline{r}_{ECI}}{\partial \underline{r}_{MCI}} = \frac{\partial(\underline{r}_{MCI} + \underline{r}_M)}{\partial \underline{r}_{MCI}} = I$$

$$\frac{\partial \dot{\underline{r}}_{ECI}}{\partial \dot{\underline{r}}_{MCI}} = \frac{\partial(\dot{\underline{r}}_{MCI} + \dot{\underline{r}}_M)}{\partial \dot{\underline{r}}_{MCI}} = I$$

I = identity matrix

$\underline{r}_M$ = ECI position vector of the Moon

m^j_c = computed j^{th} measurement

p_i = i^{th} equation-of-motion parameter

3.4.1 Trajectory Partials

In TRACE66, trajectory partials for equation-of-motion-dependent parameters are obtained by numerically integrating the vector variational equation

$$\frac{\partial \ddot{\underline{r}}}{\partial \gamma} = \left[\frac{\partial \ddot{\underline{r}}_0}{\partial \underline{r}} + \frac{\partial \ddot{\underline{r}}_1}{\partial \underline{r}} + \frac{\partial \ddot{\underline{r}}_2}{\partial \underline{r}}\right] \frac{\partial \underline{r}}{\partial \gamma} + \frac{\partial \ddot{r}(\underline{r}, \dot{\underline{r}}, \gamma)}{\partial \gamma} \tag{30}$$

where γ is some parameter of the parameter vector $\underline{\gamma}$. Thus

$$\frac{\partial \underline{r}}{\partial \gamma} \leftarrow \int \frac{\partial \dot{\underline{r}}}{\partial \gamma} \leftarrow \int \frac{\partial \ddot{\underline{r}}}{\partial \gamma}$$

The equation-of-motion parameters available in the Lunar Analyzer are

Vehicle State[1]	-	$\{(\underline{r}, \dot{\underline{r}})_o, t_o\}$
Point Mass	-	$\{\mu_i, \rho_i, \Phi_i, \lambda_i\}$
Disk Mass	-	$\{\mu_i\}$
Central Body	-	$\{\mu, C_{nm}, S_{nm}\}$

[1] Initial vehicle state parameters may be expressed in various coordinate systems (Ref. 3).

3.4.2 Geometric Measurement Partials

The observation measurements used in the determination of the lunar gravity field consist exclusively of JPL/NASA Deep Space Network long-count two-way doppler measurements, denoted by the symbol CC3. Section 4 presents a detailed description of the measurement model and the equations used to compute the geometric measurement partials, $\partial CC3/\partial \underline{r}$ and $\partial CC3/\partial \dot{\underline{r}}$. Here it suffices to say that these partial derivatives are independent of the number and nature of the parameters. They are computed based only on the position and velocity of the satellite and the earth-based sensors at the appropriate times.

4. MEASUREMENT MODEL

Observation measurements used in the determination of the lunar gravity field are the JPL/NASA Deep Space Network long-count two-way doppler (see Ref. 9). A stable oscillator (see Figure 4) is used to generate a reference signal that is transmitted to the spacecraft. The reference signal frequency is subtracted from the return signal frequency, which results in the doppler tone. The two-way doppler data constitute an accumulative counter reading value; the integrated cycle count divided by its time tag difference, in seconds, becomes the two-way doppler observable. The corresponding time tag for the observable is the time halfway between the counter reading time tags.

When one performs the computations described in this section, it is necessary to calculate position and velocity of the spacecraft relative to the earth at times that may not correspond exactly to an integration time. Therefore, it is necessary to interpolate as well as to translate in order to obtain the desired results. The Appendix contains some notes on the importance of the order and the accuracy of these computations.

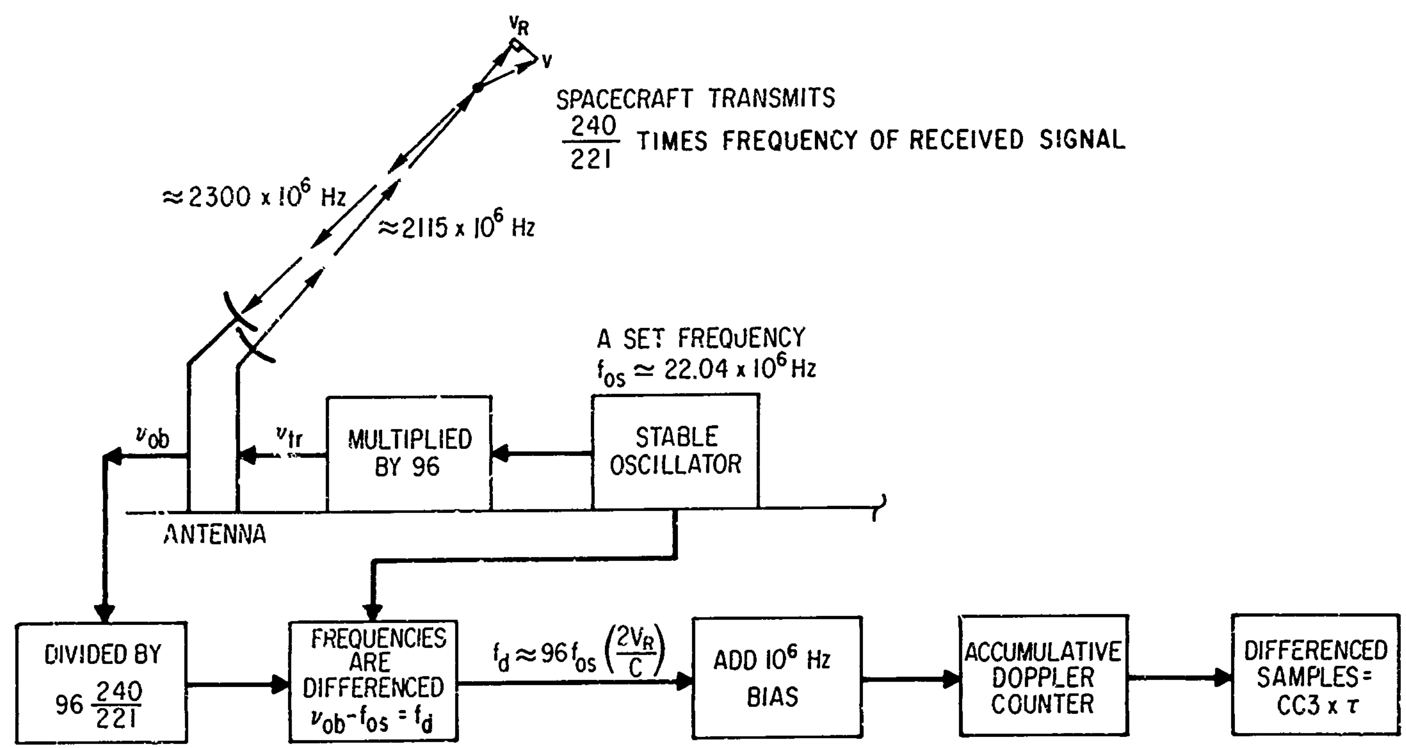

Figure 4. Simplified Two-Way Doppler System

4.1 LONG-COUNT TWO-WAY DOPPLER FREQUENCY

The computed two-way doppler observable (see Ref. 9) is expressed as

$$CC3 = \frac{f_T}{ct_C} (104.25339367) \Delta\rho \tag{31}$$

where

$f_T \equiv$ transmitter frequency $\simeq$ 22,045,600 Hz

$c \equiv$ velocity of light $\simeq$ 299792.5 km/sec

$t_C \equiv$ count time, usually 60 sec

$\Delta\rho \equiv$ topocentric range difference

Geometry for the long-count two-way doppler frequency is shown in Figure 5.

4.2 COMPUTATIONAL ALGORITHM FOR LONG-COUNT DOPPLER FREQUENCY

The notation used in the formulation for long-count doppler frequency is

$t_B \equiv$ time at which frequency count begins at the receiver

$t_E \equiv$ time at which frequency count ends at the receiver

$\underline{R}_B, \underline{R}_E$ = geocentric position vector of receiver R at time t_B and t_E, respectively; that is, $\underline{R}_B = \underline{R}(t_B)$, $\underline{R}_E = \underline{R}(t_E)$, where $\underline{R}$ is the position vector of the receiver.

$t_{SB}, t_{SE} \equiv$ t_B and t_E, respectively, minus signal transit time from spacecraft to receiver

$\underline{r}_B, \underline{r}_E \equiv$ geocentric position vector of spacecraft at time t_{SB} and t_{SE}, respectively

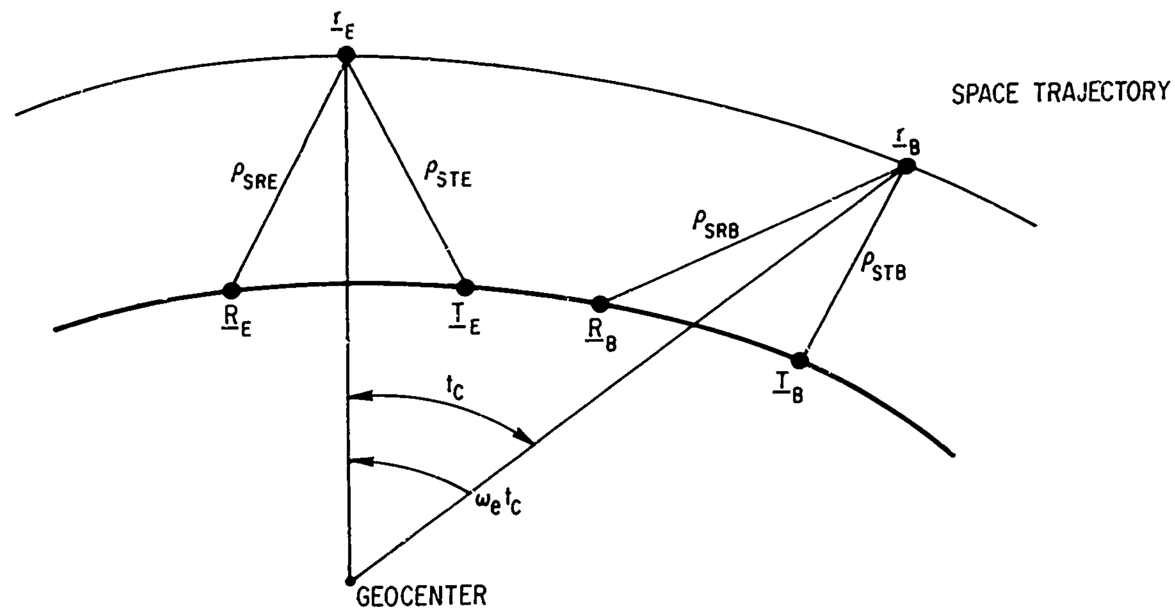

Figure 5. Long-Count Doppler Geometry

$\underline{T}_B, \underline{T}_E$ ≡ geocentric position vector of transmitter T associated with beginning and end of count, respectively

t_C ≡ total count time = $t_E - t_B$

t ≡ time tag of data on JPL B2 tapes (see Ref. 10)

ω_e ≡ Earth's angular rate

$\rho_{(\ )}$ ≡ topocentric range.

The computational algorithm consists of the following steps.

<u>Step 1.</u> t_B and t_E are obtained from t and t_C by

$$t_B = t - \frac{t_C}{2} \qquad t_E = t + \frac{t_C}{2} \tag{32}$$

<u>Step 2.</u> From t_B and t_E, $\underline{R}_B$ and $\underline{R}_E$ may be found directly by known formulas.

<u>Step 3.</u> $\underline{r}_B$ is obtained by an iterative process which takes into account signal-transit-time delay.

a. It is assumed that the geocentric position $\underline{r}$ of the spacecraft has been integrated from the equations of motion and is a known function of time that may be interpolated to the required accuracy at any given epoch.

b. First estimates of the transit time Δt_{SRB} along the SRB leg (Figure 5) and the time t_{SB} are given by

$$\Delta t^1_{SRB} = \frac{|\underline{r}_B(t_B)|}{c}$$

$$t^1_{SB} = t_B - \Delta t^1_{SRB} \tag{33}$$

c. The initial estimate of the range ρ^1_{SRB} is

$$\rho^1_{SRB} = |\underline{r}_B(t^1_{SB}) - \underline{R}(t_B)| + \Delta\rho_{SRB} \tag{34}$$

where $\Delta\rho_{SRB}$ is a refraction correction calculated once and held fixed for succeeding iterations.

d. An improved estimate of transit time is

$$\Delta t^2_{SRB} = \frac{\rho^1_{SRB}}{c} \tag{35}$$

from which is obtained

$$t^2_{SB} = t_B - \Delta t^2_{SRB} \tag{36}$$

e. Step 3.c is repeated to find that

$$\rho^2_{SRB} = |\underline{r}_B(t^2_{SB}) - \underline{R}(t_B)| + \Delta\rho_{SRB} \tag{37}$$

f. Convergence is obtained if

$$\left|\rho_{SRB}\left(t_{SB}^{n}\right) - \rho_{SRB}\left(t_{SB}^{n-1}\right)\right| < 10^{-7}\text{km}$$

Otherwise return to Step 3.d.

Step 4. After t_{SB}^{n} and $\underline{R}\left(t_{SB}^{n}\right)$ are found, $\underline{T}_B$ is determined. Thus, by a procedure similar to the above and by starting with $t_{TB} = t_{SB}^{n} - \Delta t_{SRB}^{n}$,

$$\rho_{STB} = \left|\underline{r}\left(t_{SB}^{n}\right) - \underline{T}_B\left(t_{TB}\right)\right| \tag{38}$$

where both quantities on the right side are obtained from the last iteration of Step 3. The first estimate of ρ_{STB} is then

$$\rho_{STB} = \left|r\left(t_{SB}^{n}\right) - \underline{T}_B\left(t_{TB}\right)\right| + \Delta\rho_{STB}$$

$$\Delta t_{STB} = \frac{\rho_{STB}}{c} \tag{39}$$

and $\Delta\rho_{STB}$ is the refraction correction

$$t_{TB}^{2} = t_{SB} - \Delta t_{STB}$$

$$\rho_{STB}^{2} = \left|\underline{r}\left(t_{SB}\right) - \underline{T}_B\left(t_{TB}^{2}\right)\right| + \Delta\rho_{STB} \tag{40}$$

Step 5. Steps 3 and 4 are repeated to solve for ρ_{SRE} and ρ_{STE}. For successive observations separated by a time interval t_C, ρ_{SRE} and ρ_{STE} of the m^{th} observation are the same as

ρ_{SRB} and ρ_{STB}, respectively, of the $(m + 1)^{st}$ observation. For observations separated by more than t_C, the calculations of ρ must be made for all four legs.

Step 6. The refraction correction for each leg is computed from

$$\Delta\rho = \frac{.0018958N}{340} (\sin E + .06483)^{-1.4}$$

where E is the elevation angle of the receiver (or transmitter) on the appropriate leg at t_{SB} (or t_{TB}, etc., depending on the leg); and N is the surface refractivity in units of 10^{-6}, nominally equal to 300. The correction is added to each calculated $\rho_{(\ )}$.

Step 7. Form the quantity $\Delta\rho$ (including refraction corrections) as

$$\Delta\rho = \rho_{STE} + \rho_{SRE} - \rho_{STB} - \rho_{SRB}$$

Step 8. Calculate a two-way doppler from

$$CC3 = \frac{f_T}{ct_C} (104.25339367)\, \Delta\rho \tag{41}$$

4.3 MEASUREMENT PARTIAL DERIVATIVES

The measurement partial derivatives $\partial CC3/\partial p$ are calculated from the chain rule and simplifying approximations

$$\rho_{STE} - \rho_{STB} \approx \dot{\rho}_{STE} t_C$$

$$\rho_{SRE} - \rho_{SRB} \approx \dot{\rho}_{SRE} t_C$$

where

$$(\rho\ \dot{\rho})_{STE} = (\underline{r}_E - \underline{T}_E)\cdot(\dot{\underline{r}}_E - \dot{\underline{T}}_E)$$

$$(\rho\ \dot{\rho})_{SRE} = (\underline{r}_E - \underline{R}_E)\cdot(\dot{\underline{r}}_E - \dot{\underline{R}}_E)$$

Thus, Eq. (41) becomes

$$CC3 = B[\ \dot{\rho}_{STE} + \dot{\rho}_{SRE}] \tag{42}$$

where

$$B = \frac{f_T}{c}\ (104.25339367)$$

Let the components of the vector $\underline{r}$, $\underline{R}$, $\underline{T}$ be expressed as

$$\underline{r} = (x, y, z)^T$$

$$\underline{R} = (x_R, y_R, z_R)^T$$

$$\underline{T} = (x_T, y_T, z_T)^T$$

then

$$\frac{\partial CC3}{\partial x} = B\left[\left(\frac{\dot{x} + \omega_e y_T - \dot{\rho}\ L_x}{\rho}\right)_{STE} + \left(\frac{\dot{x} + \omega_e y_R - \dot{\rho}\ L_x}{\rho}\right)_{SRE}\right] \tag{43}$$

In Eq. (43), the expression $(\ )_{STE}$ means that all quantities inside the parentheses are evaluated along the leg STE; and L_x is a direction cosine defined in

$$(L)_{STE} = (L_x, L_y, L_z)_{STE} = \left(\frac{x - x_T}{\rho}\ ,\ \frac{y - y_T}{\rho}\ ,\ \frac{z - z_T}{\rho}\right)_{STE} \tag{44}$$

Also,

$$\frac{\partial CC3}{\partial y} = B\left[\left(\frac{\dot{y} - \omega_e x_T - \dot{\rho}\, L_y}{\rho}\right)_{STE} + \left(\frac{\dot{y} - \omega_e x_R - \dot{\rho}\, L_y}{\rho}\right)_{SRE}\right]$$

$$\frac{\partial CC3}{\partial z} = B\left[\left(\frac{\dot{z} - \dot{\rho}\, L_z}{\rho}\right)_{STE} + \left(\frac{\dot{z} - \dot{\rho}\, L_z}{\rho}\right)_{SRE}\right]$$

$$\frac{\partial CC3}{\partial \dot{x}} = B\left[(L_x)_{STE} + (L_x)_{SRE}\right]$$

$$\frac{\partial CC3}{\partial \dot{y}} = B\left[(L_y)_{STE} + (L_y)_{SRE}\right]$$

$$\frac{\partial CC3}{\partial \dot{z}} = B\left[(L_z)_{STE} + (L_z)_{SRE}\right] \tag{45}$$

Now let p be any parameter to be estimated, and $\underline{S} = (\underline{r}, \dot{\underline{r}})$ where $\underline{r} = (x, y, z)^T$ and $\dot{\underline{r}} = (\dot{x}, \dot{y}, \dot{z})^T$. Then the 6×1 matrix of solutions to the variational equations $\partial \underline{S} / \partial p = \partial(x, y, z, \dot{x}, \dot{y}, \dot{z}) / \partial p$ is obtained by integrating a set of differential equations, as described in Section 3. Therefore, the required measurement partial derivative needed for the differential correction is

$$\frac{\partial CC3}{\partial p} = \left(\frac{\partial CC3}{\partial \underline{S}}\right)\left(\frac{\partial \underline{S}}{\partial p}\right) \tag{46}$$

5. NORMAL MATRIX ACCUMULATION

5.1 NORMAL MATRIX ACCUMULATION PER ARC

The augmented normal matrix is a weighted matrix product of the form A^TWA, where A is the matrix of measurement partial derivatives and W is the inverse of the measurement error variance-covariance matrix. The j^{th} row of the A matrix has the form

$$\frac{\partial m_c^j}{\partial P_1}\ , \quad \frac{\partial m_c^j}{\partial P_2}\ , \quad \ldots , \quad \frac{\partial m_c^j}{\partial P_n}\ , \quad O_{mc}^j$$

where

m_c^j = the computed j^{th} measurement

P_i = the i^{th} parameter, $i = 1, \ldots, n$

$\frac{\partial m_c^j}{\partial P_i}$ = the partial derivative of the j^{th} measurement with respect to the i^{th} parameter

O_{mc}^j = the j^{th} measurement residual; $O_{mc}^j = m^j - m_c^j$, where m^j is the actual j^{th} measurement

Let O_c denote the vector of computed measurements. Then, when the parameters are quantities appearing in the equations of motion (including initial conditions), the measurement partial derivatives are computed from the chain rule

$$\frac{\partial O_c}{\partial P} = \frac{\partial O_c}{\partial \underline{r}} \frac{\partial \underline{r}}{\partial P} + \frac{\partial O_c}{\partial \dot{\underline{r}}} \frac{\partial \dot{\underline{r}}}{\partial P} \qquad (47)$$

where $\partial \underline{r}/\partial P$ and $\partial \dot{\underline{r}}/\partial P$ are referred to as trajectory partials, and are solutions of the variational equations. The matrices $\partial O_c/\partial \underline{r}$ and $\partial O_c/\partial \dot{\underline{r}}$, and the

columns of A that correspond to parameters not appearing in the equations of motion (station locations and measurement error terms) are referred to as the geometric partials and are computed analytically.

If W were a full matrix, it would be necessary to compute the entire A matrix at one time in order to perform the matrix multiplications. However, in the Lunar Gravity Analysis, it is assumed that all observations are independent and have the same variance. Therefore, W is of the form

$$W = 1/\sigma^2 \, I$$

where I is an identity matrix. The normal matrix may be written as

$$\begin{aligned} A^T W A &= 1/\sigma^2 \, A^T A \\ &= 1/\sigma^2 \sum_{i=1}^{n} [a_{ij} \, a_{ik}] \end{aligned} \tag{48}$$

where

$$a_{ij} = \text{the } i - j^{th} \text{ element of the matrix A}$$

$$1/\sigma^2 \sum_{i=1}^{n} a_{ij} \, a_{ik} = \text{the } j - k^{th} \text{ element of the matrix } A^T W A$$

The first n columns of the $A^T WA$ comprise the normal matrix of the least-squares problem. The last column is the right-hand side of the system $A^T W O_{mc}$ with last element of that column equal to the weighted sum of the squares of the residuals $O_{mc}^T W \, O_{mc}$. Note that by using Eq. (48), the A matrix needs to be formed only one row at a time, and the contribution from that particular observation is summed into the $A^T WA$. Furthermore,

since the normal matrix is symmetric, the accumulation is done only for $k \geq j$, thereby eliminating the computations for all elements below the principal diagonal.

5.2 MERGING THE SINGLE-ARC MATRICES

A computer program external to TRACE66 takes the single-arc normal matrices generated as described above and merges them into one large linear system. All input and output matrices of the merge procedure are stored on magnetic tape. A priori inverse variances may be added to the diagonal elements corresponding to mass parameter magnitudes at the time of the merge. Vehicle state parameter a prioris can be added if desired at the time of the single-arc matrix accumulation in TRACE66. The form of the final merged normal matrix is shown in Figure 6.

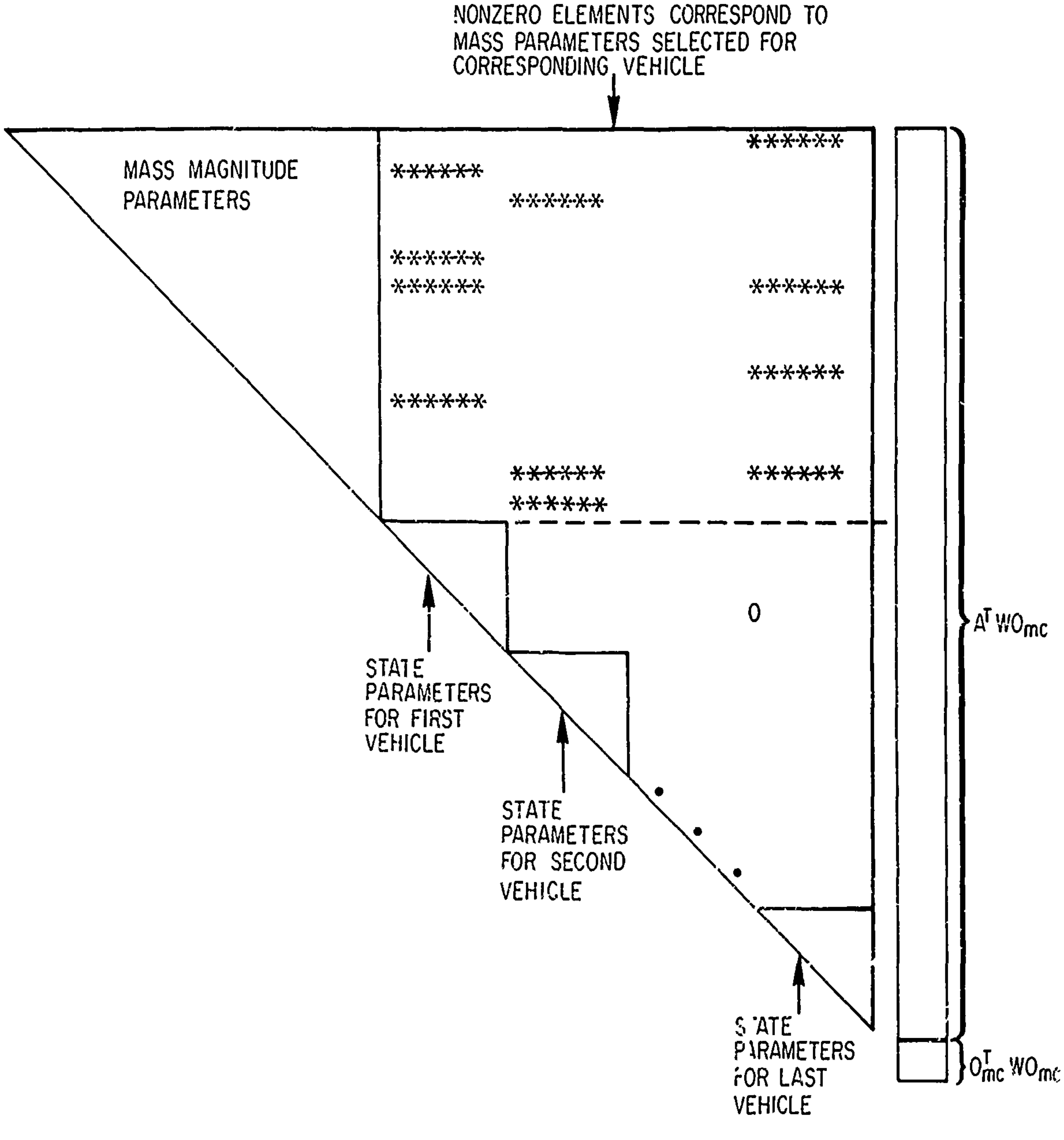

Figure 6. Form of the Merged Normal Matrix

6. SOLUTION OF THE NORMAL EQUATIONS

Two alternate techniques are used to solve the normal equations. The first method, for problems with a limited parameter set (i.e., a maximum of 100 parameters), computes the inverse of the normal matrix as well as the parameter corrections. In addition, a side condition can be imposed that bounds the magnitude of the parameter corrections. This capability, including the solution, is completely self-contained in TRACE66.

The second method is used for problems with an effectively unlimited number of parameters. It uses mass storage devices during the solution process, and therefore does not require the retention of the entire normal matrix within core. With this method the solution of the normal equations is performed in a program exterior to TRACE66. (Note that the inverse of the normal matrix is not computed in this method.)

6.1 SOLUTION FOR A LIMITED PARAMETER SET

The complete solution procedure, including the bounding feature, is fully discussed in Reference 11, and will be only briefly described here. The reference states that if a function to minimize is given, as well as a bounding condition of the form

$$f(\Delta P) = \left(A\Delta P - O_{mc}\right)^T W(A\Delta P - O_{mc}) \tag{49}$$

and

$$(G\Delta P)^T (G\Delta P) \leq 1$$

the value of ΔP, which minimizes f subject to the constraint, is the solution of the linear system

$$(A^T WA + ZG^T G)\,\Delta P = A^T W\, O_{mc} \tag{50}$$

where Z is a Lagrange multiplier. The proper value of Z may be determined by an iterative process. Therefore, the problem is always reduced to the finding of the solution to a system of the form

$$CX = B$$

where C is an $n \times n$ positive definite symmetric matrix; X is an m vector of unknowns; and B is an m vector referred to as the right-hand side. The procedure used to compute X and invert the matrix C is finite (noniterative) and applicable only to symmetric matrices.

It is based upon the fact that a symmetric matrix can be decomposed into a product of the form: $C = LDL^T$ where L is lower-triangular with 1 on

the diagonal; and D is a diagonal matrix. In such a representation, $\det(L) = \pm 1$, and $\det(D) = \det(C)$. Therefore, L^{-1} exists, and is also lower-triangular with -1 on the diagonal; and D has no zero diagonal element if C is non-singular. Thus

$$D = L^{-1} C (L^T)^{-1} = SCS^T$$

$$C^{-1} = (L^T)^{-1} D^{-1} L^{-1} = S^T D^{-1} S \tag{51}$$

where $S = L^{-1}$.

The procedure builds S one row at a time with a bordering technique. Since D is diagonal, once S is obtained, the calculations of X and C^{-1} are reduced to matrix multiples.

6.2 SOLUTION FOR AN UNLIMITED PARAMETER SET

When a great many unknowns are being considered, the system is solved by using a technique based upon the Gaussian elimination method. The procedure is to successively eliminate variables from the system until the problem is reduced to one equation with one unknown. By using the solution of this equation as a starting point, values for the remaining unknowns are obtained through back substitution.

Symbolically, the process is expressed as follows: Let an $n \times n$ system be of the form $CX = B$, where C is an $n \times n$ matrix $[c_{ij}]$, X an m vector of unknowns $[x_i]$, and B an m vector $[b_i]$. Then eliminate the k^{th} column and row ($k = 1, \cdots n - 1$) by modifying all c_{ij} and b_i, where $i, j > k$ in the manner

$$c'_{ij} = c_{ij} - \frac{c_{ik}\, c_{kj}}{c_{kk}}$$

$$b'_i = b_i - \frac{c_{ik}\, b_k}{c_{kk}} \tag{52}$$

Finally, after (n - 1) eliminations, the system will be reduced to

$$c_{nn}\, x_n = b_n$$

where c_{nn} and b_n have been modified (m - 1) times. Then,

$$x_n = b_n / c_{nn}$$

Using the solutions for $x_{k+1}, \cdots, x_n$, produces the solution for x_k, which is

$$x_k = \left(b_k - \sum_{j=k+1}^{n} c_{kj}\, x_j \right) \Bigg/ c_{kk} \tag{53}$$

where the c's and b's are the modified coefficients after all eliminations.

A slight modification of the technique can be used to take advantage of a symmetric matrix. As the symmetry property can be expressed as $c_{ij} = c_{ji}$ for all i and j, the basic modification equation can be rewritten as

$$c'_{ij} = c_{ij} - \frac{c_{ik}\, c_{jk}}{c_{kk}} \tag{54}$$

The above needs to be calculated only for all $j > k$, and for all $i \geq j$ for each j. This eliminates the computation of all modified elements above the principal diagonal of the matrix.

Another sophistication of the technique significantly increases the algorithm's efficiency. It reduces the number of multiples required when the matrix is sparse with strings of consecutive zeros in some columns. When c_{ik} is zero, none of the elements in the i^{th} row need be modified for eliminating the k^{th} unknown. By using an indexing scheme that points to the beginning and end of strings of zero elements in the matrix, computation time is considerably reduced.

For systems of the size being considered, a large number of central memory words of data storage would be required to obtain the solution in core. For example, a system on the order of 2000 would require in excess of 2, 000, 000 words. Because this amount of core storage is not feasible, a technique of file usage (either magnetic tape or disk) was devised to take advantage of mass storage devices. Three files are used with the following designations and purposes:

ATAPE	Contains the matrix C stored as lower triangular by columns
TTAPE	Contains the transformations necessary for eliminating each column, along with start/stop pointers for strings of zero elements in that column
NBACK	Contains the transformed A matrix to be used for back substitution.

A buffer of the maximum available size is provided for the program. The number of file manipulations required is inversely proportional to the size of this buffer. Therefore, an increase in buffer size results in a reduction of job throughput time. The right-hand side (B vector) is kept in core. The buffer is filled from ATAPE with as many columns of C as will fit.

Starting with the first column of C, which is in the buffer, and working sequentially towards the last column in it, transformations are computed; applied to the remaining columns in the buffer and to the B vector; and then written on TTAPE. When the last column has been processed, the contents of the buffer are written on NBACK. The buffer is refilled, TTAPE is rewound, and the transformations are read in and applied to all columns. The process is then repeated for this buffer load, starting with the computation of the transformation for the first column. (Note that additional transformations are added to TTAPE for every buffer load.) After the last column of the matrix has been processed, back substitution is begun. The necessary transformed C matrix is obtained by initially reading the last buffer load from NBACK, processing it, backspacing NBACK twice, reading the next-to-last buffer load, and so on, until the full solution is obtained.

The total execution time depends upon the order of the system being solved, the size of the buffer supplied, and the type of mass storage devices being used. Central processor (CP) time, however, is only slightly affected by buffer size or storage device. The following empirical equation (although slightly pessimistic for large systems, $n > 1000$), has proved reliable for typical problems:

$$CP \simeq 0.2 + 0.0075\, n + .00003\, n^2 + 1.77 \times 10^{-7}\, n^3 \text{ (sec)} \qquad (55)$$

As successive columns of the matrix are eliminated, divisions are performed by the current diagonal element of that column (i.e., the one transformed by all previous eliminations). Because the matrix is symmetric and positive definite, all transformed diagonals are smaller than the original diagonal element. A comparison of the element used for the division with the original diagonal element provides information regarding the conditioning of the matrix. If the ratio of these numbers is less than some ϵ (for example, $\epsilon = 10^{-5}$), then the corresponding unknown is said to be poorly determined.

On option, that unknown may be, in effect, eliminated from the system by zeroing out the entire row and column, and setting the diagonal element to a large number (for example, 10^{100}). Such a procedure enhances the stability of the system. In the case of the Lunar Gravity Field determination, the solution is essentially unaltered for a small number of parameter eliminations.

7. COMPOSITE LUNAR GRAVITY FIELD

Several representations of the lunar gravity field have been obtained recently (see Ref. 12). The fields differ more in the methods of analysis than in the basic data, since all the results were based on the doppler tracking of Lunar Orbiters. A procedure for combining a mascon gravity model (see R ` 13) with spherical harmonics to obtain a composite model of the lunar gravity field is described in Sections 8.1 and 8.2.

7.1 SYNTHESIS OF THE COMPOSITE FIELD

Let the surface of a sphere of radius r be divided into two regions (see Figure 7). Region R_1 is the one under which mascons have been obtained, and R_2 is the remainder of the sphere. Now let a network, or grid of points on the sphere, have coordinates (r, Φ_j, Λ_j) at the j^{th} point. Let j_1 be the number of points in R_1, and j_2 the number in R_2; and J the total number, or $J = j_1 + j_2$. At each of the j_1 points in R_1, the vertical component of the gravity disturbance g_r is calculated from the mascon distribution, while in R_2, g_r is calculated from a spherical harmonics model. The value of r is chosen so that the altitude is 100 km above the mean lunar surface, because most of the best doppler tracking for the mascon model occurred at this altitude.

The disturbance potential at the i^{th} grid point caused by the i^{th} mascon (see Figure 8) is given by

$$- V_{ij} = \frac{\mu_i}{\rho_{ij}} = \mu_i\left[a^2 + r^2 - 2ar\cos\psi_{ij}\right]^{-1/2} \tag{56}$$

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where

$$\cos \psi_{ij} = \sin \phi_i \sin \Phi_j + \cos \phi_i \cos \Phi_j \cos (\Lambda_j - \lambda_i)$$

$\mu_i \equiv$ magnitude of i^{th} mascon

$a, \phi_i, \lambda_i \equiv$ mean lunar radius, selenographic latitude, and longitude, respectively, of i^{th} mascon

$r, \Phi_j, \Lambda_j \equiv$ selenographic distance, latitude, and longitude of j^{th} grid point.

The corresponding radial component of gravity disturbance is

$$-\frac{\partial V_{ij}}{\partial r} = -\frac{\mu_i}{\rho_{ij}^3} (r - a \cos \psi_{ij}) \tag{57}$$

If I is the total number of mascons in the distribution and the net value of g_r is a superposition of all the contributions, then

$$g_r (r, \Phi_j, \Lambda_j) = \sum_{i=1}^{I} \left(- \frac{\partial V_{ij}}{\partial r_j} \right) = - \sum_{i}^{I} \frac{\mu_i}{\rho_{ij}^3} (r - a \cos \psi_{ij}) \tag{58}$$

Equation (58) is then used to obtain a g_r value for each of the j_1 grid points in R_1.

At each of the j_2 points in R_2, a g_r value can be calculated, from the radial derivative of the disturbance spherical harmonics work function (negative of potential) of degree N, as

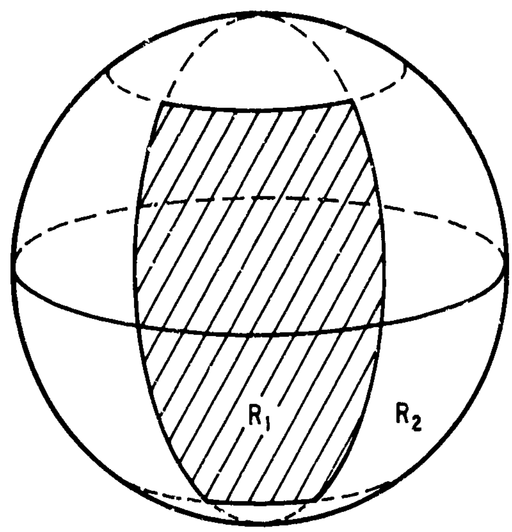

Figure 7. Geometry of Surface Regions

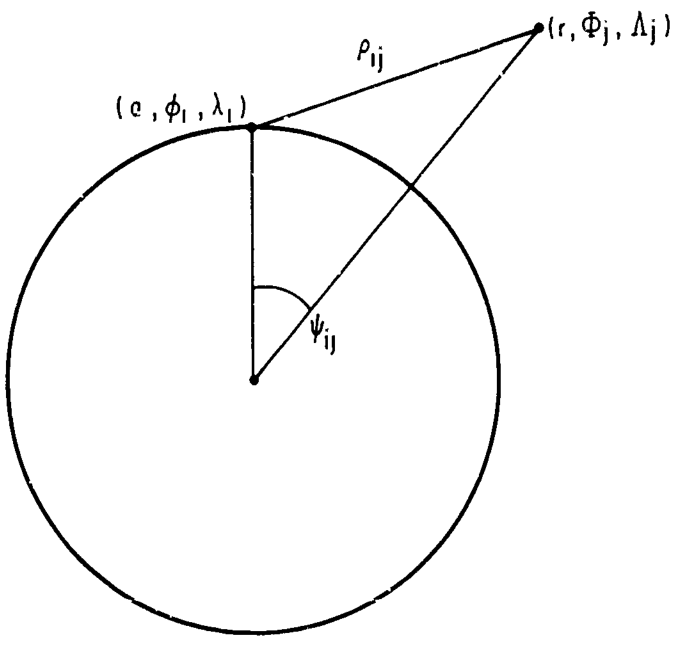

Figure 8. Geometry of Grid Point

$$V(r, \Phi, \Lambda) = \frac{\mu_m}{a} \sum_{n=0}^{N} \sum_{m=0}^{n} \left(\frac{r}{a}\right)^{n+1} P_{nm} (\sin \Phi) (\bar{C}_{nm} \cos m\Lambda + \bar{S}_{nm} \sin m\Lambda)$$

$$= g_o\, a \sum_{n} \sum_{m} \left(\frac{r}{a}\right)^{n+1} (\bar{C}_{nm} Y^c_{nm} + \bar{S}_{nm} Y^s_{nm}) \qquad (59)$$

where

$\mu_m \equiv$ lunar mass

$r, \Phi, \Lambda \equiv$ spherical coordinates of grid point in R_2

$\bar{C}_{nm}, \bar{S}_{nm} \equiv$ normalized coefficients

$P_{nm} \equiv$ associated Lengendre polynomial (fully normalized)

$g_o \equiv \mu_m/a^2$

The radial component of gravity disturbance being

$$g_r(r, \Phi, \Lambda) = - g_o \sum_{n} \sum_{m} \left(\frac{a}{r}\right)^{n+2} (n + 1) (\bar{C}_{nm} Y^c_{nm} + \bar{S}_{nm} Y^s_{nm}) \qquad (60)$$

a g_r value for each point in R_2 can then be calculated from Eq. (60).

7.2 HARMONIC ANALYSIS

A harmonic analysis of the radial gravity field can now be obtained from Eqs. (58) and (60). Let the expansion of the composite field g_r^c have the form of Eq. (60); that is

$$g_r^c(r, \Phi, \Lambda) = -g_o \sum_n^p \sum_m \left(\frac{a}{r}\right)^{n+2} (n+1)\,(\bar{C}_{nm}\, Y_{nm}^c + \bar{S}_{nm}\, Y_{nm}^s) \tag{61}$$

where the symbols have similar meanings and p is the degree of the new expansion. If all of the grid values of g_r (denoted g_r^o) are assigned equal weights, the coefficients in the expansion of Eq. (61) are easily obtained from the orthogonality properties of Legendre polynomials, so that

$$(n+1)\begin{pmatrix}\bar{C}_{nm}\\ \bar{S}_{nm}\end{pmatrix} = -\frac{1}{g_o}\left(\frac{r}{a}\right)^{n+2} \frac{1}{4\pi}\int_{\text{Sphere}} g_r^o\,(\Phi, \Lambda)\begin{pmatrix}Y_{nm}^c\\ Y_{nm}^s\end{pmatrix} d\Omega$$

$$\cong -\frac{1}{g_o}\left(\frac{r}{a}\right)^{n+2} \frac{1}{J}\sum_{j=1}^{J}\left[g_r^o\,(\Phi, \Lambda)\begin{pmatrix}Y_{nm}^c\\ Y_{nm}^s\end{pmatrix}\right]_j \tag{62}$$

7.3 AUXILIARY CALCULATIONS

After the harmonic analysis, a number of auxiliary calculations are obtainable, such as the following.

a. An RMS fit over regions R_1 and R_2 is obtained by

$$RMS_1 = \left[\sum_{j=1}^{j_1} W_j\,(g_r^o - g_r^c)_j^2\right]^{1/2} j_1^{-1/2}$$

$$RMS_2 = \left[\sum_{j=1}^{j_2} W_j\,(g_r^o - g_r^c)_j^2\right]^{1/2} j_2^{-1/2} \tag{63}$$

$$RMS = \left[RMS_1^2 + RMS_2^2\right]^{1/2} \quad \text{the RMS power in the field}$$

b. For $r = a$, the power spectrum, or degree variance, of the potential coefficients is

$$RMS_t = \left[\sum_j (g_r^o)^2\right]^{1/2} J^{-1/2} \tag{64}$$

c. The spectrum of vertical gravity at $r = a + h$, where $h = 100$ km, is given by

$$\sigma_n^2 = \sum_m \left(\bar{C}_{nm}^2 + \bar{S}_{nm}^2\right) \tag{65}$$

d. The potential coefficients $\bar{C}_{nm}$, $\bar{S}_{nm}$ and their unnormalized equivalents are obtained by

$$\Sigma_n = g_o \left[\sum_m \left(\frac{a}{r}\right)^{n+4} (n+1)^2 \left(\bar{C}_{nm}^2 + \bar{S}_{nm}^2\right)\right]^{1/2} \tag{66}$$

7.4 REMARKS

The procedure described above for obtaining a composite lunar gravity field is similar to the one applied to the harmonic analysis of terrestrial gravimetry. In the latter, measured surface gravity is assigned to squares with measurements, and gravity from a satellite field is assigned to squares without measurements. However, the basic data base is completely different for the two cases, as the expansion obtained at 100 km can be analytically continued to the lunar surface since there are no intervening masses.

The maximum number for J should be about 1640 corresponding to 5×5 deg squares, and the order of the expression should have a limit of 20, which implies the solution of a system of order 441.

The composite lunar gravity field computations are performed outside of TRACE66.

8. LUNAR GRAVITY ANALYSIS FEATURES

In addition to the normal matrix accumulation and least-squares simulation options described, the following features are available in TRACE66.

8.1 MASS PARAMETER SELECTION

The mass parameters to be estimated are selected by the following criterion: The acceleration arising from a given mass (point or disk) integrated over the counting interval must exceed the noise threshold of 0.01 Hz at some point along a given trajectory before that mass is selected for regression from the doppler data on the trajectory. The criterion can be translated into an approximate condition on the distance P from the center of the point or disk mass to the satellite trajectory; that is, P must be less than some minimum distance.

The computational algorithm for mass parameter selection is

a. $$P = |[S]^T \underline{r} - \underline{P}_i|$$

b. $$P \le (\alpha + \beta h) \tag{67}$$

where

P_i = selenographic position vector of the i^{th} mass

$[S]$ = transformation matrix from selenographic coordinates to selenocentric frame

α, β = altitude bias and scale factors, respectively

$\underline{r}$ = selenocentric position vector of the vehicle

h = vehicle altitude.

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8.2 RESIDUAL CALCULATIONS

Calculated and plotted at each observation time are: the filtering of observation data blunder points; the doppler residuals; spacecraft selenographic latitudes, longitudes, and heights.

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APPENDIX

INTERPOLATION ACCURACY IN TRACE66 WITH APPLICATION TO LUNAR EPHEMERIDES

A.1 METHOD

TRACE66 uses a Hermitian interpolation scheme to calculate positions and velocities at times not exactly equal to the time of some integration point. The Hermitian polynomial used to calculate position is a function of position, velocity, and acceleration at two different times, one on each side of the desired unknown time point. These six pieces of information allow the derivation of a fifth-order interpolating polynomial. The formula for the velocity is the derivative of the position equation, and is therefore a fourth-order polynomial.

A.2 ACCURACY OF THE METHOD

The accuracy of these polynomials depends on three things:

- The time interval or step size between the known points
- The relative position of the unknown point in this interval
- The nature of the function or dependent variable to be calculated; i.e., sizes of the function's derivatives

Note that calculation of a dependent variable in the exact middle of an interval leads to the largest interpolation error that will occur in that interval. All accuracy figures given in this Appendix correspond to interpolation at the interval's midpoint for the ephemeris of a typical low-altitude earth satellite. The results of an empirical study of interpolation accuracy versus step size are presented in Fig. A-1. The fifth-order position and fourth-order velocity curves were calculated from the equations used in TRACE66. The third-order Hermitian curves are plotted for comparison purposes. It is expected

that a smoother function such as the ephemeris of a high-altitude earth satellite would yield greater accuracy, whereas a more radical function might yield somewhat poorer accuracy.

A.3 APPLICATION TO LUNAR EPHEMERIDES

When TRACE66 is operated in the lunar mode, it may be necessary to calculate earth-based observations of a lunar satellite. Because of interpolation errors, the order in which such calculations are performed is extremely important. Consider, for example, a lunar satellite with a selenocentric radius vector of 2850 km and a geocentric radius vector of 380,000 km. Assume that it is desired to calculate a geocentric range at some time by interpolating from an integrated moon-centered trajectory with a 4-min step size on the tabulated points. One possible approach would be to transform the moon-centered coordinates to earth-coordinates using a tabulated lunar ephemeris and to interpolate in the earth-centered system. Assuming that the tabulated ephemeris is more accurate than the interpolation, Fig. A-1 indicates that the results should be good to within about 10 m. Alternatively, the interpolation could be done in the moon-centered system, before the transformation to earth-centered coordinates is performed. Figure A-1 shows that if the calculations are done in this order, the result is expected to be accurate within about 0.1 m. For the obvious reason, the latter method is employed in TRACE66.

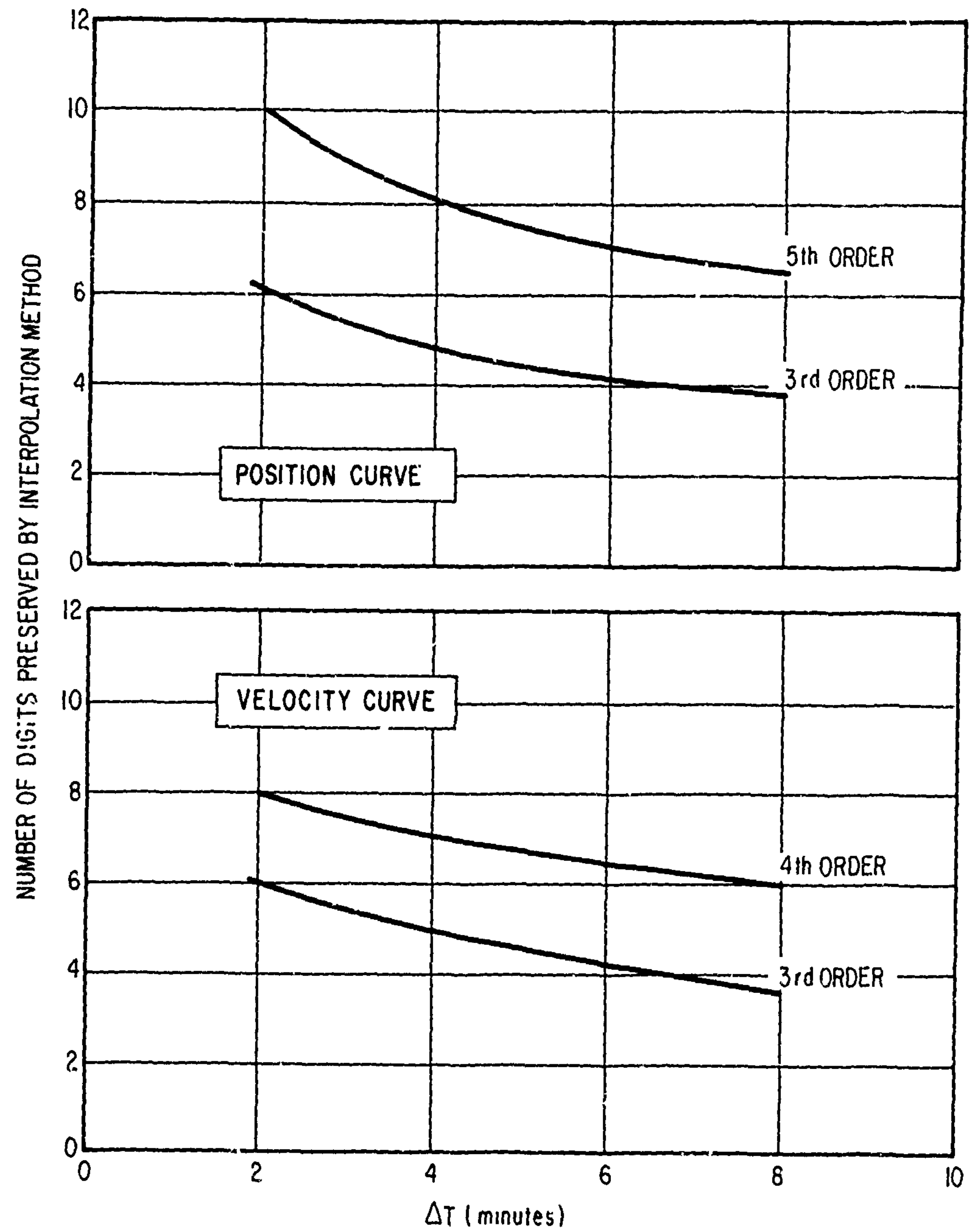

Figure A-1. Interpolation Accuracy